HW. # 3

Homework problems are taken from “Principles of Mathematical Analysis” by W. Rudin and “Real Analysis” by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If d is a metric on M, show $|d(x, z) - d(y, z)| \le d(x, y)$ for any x, y, z $\in M$.

2. As it happens, some of our requirements for a metric are redundant. To see why this is so, let M be a set and suppose that $d : M \times M \to R$ satisfies
   (a) d(x, y) = 0 if and only if x = y and
   (b) $d(x, y) \le d(x, z) + d(y, z)$ for all x, y, z $\in M$.

   Prove that d is a metric; that is, show that $d(x, y) \ge 0$ and $d(x, y) = d(y, x)$ hold for all x, y.

3. Let M be a set and suppose that $d : M \times M \to [0, \infty)$ satisfies properties (i), (ii), and (iii) for a metric on M and the triangle inequality *reversed*: $d(x, y) \ge d(x, z) + d(y, z)$. Prove that M has at most one point.

4. Let $d : M \times M \to [0, \infty)$ be a metric function on the set M. Show that $p : M \times M \to [0, \infty)$ defined by p(x, y) = min{d(x, y), 1} is also a metric function on M.

5. If d_1 and d_2 are both metrics on the same set M, which of the following yield metrics on M: $d_1 + d_2$? $\max\{d_1, d_2\}$? $\min\{d_1, d_2\}$? If d is a metric, is d^2 a metric?

6. Which of the following functions define a metric on R?
   (a) $d_1(x, y) = |x^7 - y^7|$
   (b) $d_2(x, y) = |x - y|^3$
   (c) $d_3(x, y) = |x - y|^{2/3}$
   (d) $d_4(x, y) = \min\{\sqrt{|x - y|}, 1\}$
   (e) $d_5(x, y) = \sqrt{|x - y|} + \ln\left(\frac{|x - y|}{1 + |x - y|} + 1\right)$

7. Let $0 < \alpha < 1$. Show that if x and y are positive real numbers, then $\left|x^{\alpha} - y^{\alpha}\right| \le |x - y|^{\alpha}$. In particular, $\left|\sqrt{x} - \sqrt{y}\right| \le \sqrt{|x - y|}$. [Hint: Prove that $d(x, y) = |x - y|^{\alpha}$ defines a metric on R and use exercise 1]

8. Let R^{∞} denote the collection of all real sequences $x = \{x_n\}$. Show that the expression

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

defines a metric on R^{∞}. Can you think of other metrics?

9. Check that $d(f, g) = \max_{a \le t \le b} |f(t) - g(t)|$, $p(f, g) = \int_a^b |f(t) - g(t)| dt$, and $\sigma(f, g) = \int_a^b \min\{|f(t) - g(t)|, 1\} dt$ define metrics on C[a, b], the vector space of real-valued continuous functions over the closed interval [a, b].

10. We say that a subset A of a metric space M is **bounded** if there is some $x_0 \in M$ and some constant $C < \infty$ such that $d(a, x_0) \le C$ for all $a \in A$. Show that a finite union of bounded sets is again bounded.

11. We define the **diameter** of a nonempty subset A of M by $diam(A) = \sup\{d(a, b) : a, b \in A\}$. Show that A is bounded if and only if diam(A) is finite.

12. Show that $\|x\|_{\infty} \le \|x\|_2 \le \|x\|_1$ for any $x \in R^n$. Also check that $\|x\|_1 \le n\|x\|_{\infty}$ and $\|x\|_1 \le \sqrt{n}\|x\|_2$.

13. Show that diam($B_r(x)$)≤ 2r, and give an example where strict inequality occurs.

14. If diam(A) < r, show that $A \subset B_r(a)$ for some $a \in A$.

15. If $A \subset B$, show that diam(A) ≤ diam(B)

16. Give an example where diam($A \cup B$) > diam(A) + diam(B). If $A \cap B \ne \emptyset$, show that diam($A \cup B$)≤ diam(A) + diam(B).